THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixtieth regular meeting of the Society was held in New York City on Saturday, October 26, 1912, extending through the usual morning and afternoon sessions. The following fifty-two members were present:

Professor R. C. Archibald, Professor M. J. Babb, Dr. F. W. Beal, Mr. A. A. Bennett, Professor W. J. Berry, Professor G. D. Birkhoff, Dr. Henry Blumberg, Professor Joseph Bowden, Professor E. W. Brown, Professor B. H. Camp, Dr. A. S. Chessin, Professor J. G. Coffin, Professor F. N. Cole, Dr. E. B. Cowley, Dr. H. B. Curtis, Dr. L. S. Dederick, Dr. L. L. Dines, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Professor Wilbert Garrison, Professor O. E. Glenn, Professor C. C. Grove, Professor H. E. Hawkes, Professor E. V. Huntington, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Professor James Maclay, Dr. R. L. Moore, Professor W. F. Osgood, Mrs. Anna J. Pell, Professor James Pierpont, Dr. H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Professor L. P. Siceloff, Mr. C. G. Simpson, Mr. L. L. Smail, Professor D. E. Smith, Professor P. F. Smith, Professor Elijah Swift, Professor Henry Taber, Dr. E. H. Taylor, Professor C. B. Upton, Mr. C. E. Van Orstrand, Professor Vito Volterra, Mr. H. E. Webb, Professor H. S. White, Professor A. H. Wilson.

The attendance also included Professor Emile Borel, of the University of Paris. Professors Borel and Volterra were among the foreign lecturers at the recent dedicatory exercises of the Rice Institute, Houston, Texas, and have since delivered lectures at several American universities.

Vice-President Henry Taber occupied the chair during both sessions. The Council announced the election of the following persons to membership in the Society: Dr. Henry Blumberg, Brooklyn, N. Y.; Mr. J. M. Colaw, Monterey, Va.; Dr. F. M. Morgan, Dartmouth College; Dr. Louis O'Shaughnessy, University of Pennsylvania; Dr. C. T. Sullivan, McGill University. Five applications for membership were received.

A list of nominations of officers and other members of the Council, to be placed on the official ballot for the annual meeting, was adopted. A committee was appointed to audit the Treasurer's accounts for the current year.

The following papers were read at this meeting:

(1) Dr. H. W. REDDICK: "Systems of plane curves whose intrinsic equations are analogous to the intrinsic equation of an isothermal system."

(2) Dr. L. L. DINES: "Note concerning a theorem on implicit functions."

(3) Dr. L. L. DINES: "Singular points of space curves defined as the intersections of surfaces."

(4) Dr. E. T. BELL: "On Liouville's theorems concerning certain numerical functions."

(5) Dr. E. T. BELL: "The representation of a number as a sum of squares."

(6) Mr. G. R. CLEMENTS: "Implicit functions defined by equations with vanishing Jacobian. Supplementary note."

(7) Professor EDWARD KASNER: "Note on contact transformations of space."

(8) Dr. E. H. TAYLOR: "An extension of a theorem of Painlevé."

(9) Dr. L. S. DEDERICK: "On the character of a transformation in the neighborhood of a point where its Jacobian vanishes."

(10) Professor VITO VOLTERRA: "Some integral equations."

(11) Professor W. F. OSGOOD: "Proof of the existence of functions belonging to a given automorphic group."

(12) Professor G. D. BIRKHOFF: "Proof of Poincaré's geometric theorem."

(13) Professor E. V. HUNTINGTON: "A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion."

(14) Dr. DUNHAM JACKSON: "On the degree of convergence of related Fourier series."

(15) Mr. A. A. BENNETT: "Note on the solution of linear algebraic equations in positive numbers."

In the absence of the authors the papers of Dr. Bell and Mr. Clements were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Connected with a singly infinite system of plane curves are the four intrinsic quantities T, N, T_1, and N_1. T and N are the rates of variation of the curvature of a curve of the system along the curve itself and along an orthogonal curve respectively, while T_1 and N_1 are the corresponding quantities for the orthogonal system. It is known that $T + T_1 = 0$ is the intrinsic equation of an isothermal system. In this paper Dr. Reddick considers the twelve systems whose intrinsic equations are formed by equating to zero the sums and differences of the four quantities T, N, T_1, and N_1, taken in pairs. In particular the solution of the differential equation of the family of type $T - T_1 = 0$ admitting a group of translations is found and involves elliptic integrals.

2. In the BULLETIN of last June, Mr. G. R. Clements stated (Theorem IV) a generalization of the Weierstrassian implicit function theorem. The theorem had to do with the number of solutions of a system of analytic equations

$$f_i(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0 \qquad (i = 1, 2, \cdots, p),$$

and one part of the hypothesis was

$$J_1 \equiv \frac{D(f_1, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \text{ when } (x) = 0, \quad (y) = 0,$$

. . . . . .

$$(3)\quad J_{k-1} \equiv \frac{D(J_{k-2}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \text{ when } (x) = 0, (y) = 0,$$

$$J_k \equiv \frac{D(J_{k-1}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} \neq 0 \text{ when } (x) = 0, (y) = 0.$$

Dr. Dines calls attention to the fact that the hypotheses (3) are equivalent to

$$(A)\quad \begin{array}{l} f_1 \text{ is a power series of order } k \text{ in } y_1, y_2, \cdots, y_p; \\ f_j \text{ is a power series of order 1 in } y_1, y_2, \cdots, y_p \\ \qquad\qquad\qquad\qquad\qquad\qquad (j = 2, 3, \cdots, p); \\ \text{the resultant of the ``characteristic'' polynomials of} \\ \text{the power series } f_i \text{ does not vanish.} \end{array}$$

If the assumptions (A) be substituted in place of (3), Mr. Clements' Theorem IV becomes a simple but interesting corollary of Professor Bliss's theorem published in the *Transactions* of last April.

The fact that conditions (A) imply conditions (3) furnishes a generalization of Theorem VI of Mr. Clements' paper from two equations to p equations.

3. In his second paper, Dr. Dines considers the singular points of curves defined by two equations $\phi(x, y, z) = 0$, $\psi(x, y, z) = 0$, where ϕ and ψ are real functions of real variables. If the functions can be expanded by Taylor's theorem in the forms

$$\phi(x, y, z) = \phi_m(x, y, z) + R(x, y, z),$$
$$\psi(x, y, z) = \psi_n(x, y, z) + S(x, y, z),$$

where ϕ_m and ψ_n are homogeneous polynomials of degrees m and n respctively, and R and S are the complementary remainders, then the nature of the singular point at the origin depends primarily upon the two homogeneous equations $\phi_m = 0$, $\psi_n = 0$. If ϕ_m and ψ_n have no common factor, then at most $m - n$ real branches of the curve can pass through the origin. Criteria are obtained for determining the number of real branches through the origin and for detecting the presence of cusps; and methods are exhibited for analyzing the singularity. If ϕ_m and ψ_n have a common factor of degree k, then the maximum number of real branches which can pass through the origin is $mn + k$. A method is given for reducing the investigation in this case to the solution of two equations in which the leading polynomials have no common factor. A special study is made of singular points of the second order.

4. The theorems referred to in Dr. Bell's first paper are those published by Liouville, *Journal* for 1857, four articles, and scattered papers relating to these in subsequent volumes. As stated by Liouville, the algebraic manipulations necessary for verifications are not always easy, and clearly the theorems were not found directly, but by a "simple and uniform method," etc., which it is the object of this paper to set forth. First all of the theorems are proved directly from first principles, and then all are derived with a great many more by the multiplication and transformation of two or more series of the form $\Sigma f(n)/n^{sg(\omega)}$, $(n = 1, \cdots, \infty)$, s a constant, f, g two numerical functions; that is, by means of a modified zeta series. All of Liouville's theorems are consequences of the fact that his generating function has been always chosen so

that it is factorable in the same way as Riemann's zeta, and his numerical functions are all multiplicative for relatively prime values of the variable. By varying the forms of f and g, and assigning different values to s, new numerical functions are suggested, and theorems deduced. As Liouville remarks, the subject is inexhaustible, but the general principles he used are undoubtedly these. Generalizations may be found on replacing integers by norms of algebraic integers, primes by prime ideals, etc., and using the generalized zeta of Dedekind and Hilbert in the same way. But all these generalizations are included in that wherein numbers are replaced by classes, and the symbol of multiplication by the symbol indicating the greatest class common to two classes. In this way many curious relations are found. The other type of theorem is yet more general, and deals with "a species of arithmetical elimination." Liouville gives the following: "if $n = d\delta$ be any resolution of n into factors, and $A(n)$, $G(n)$, $H(n)$, $P(n)$, $Q(n)$ any numerical functions of n satisfying $\Sigma A(d)G(\delta) = H(n)$ and $\Sigma A(d)P(\delta) = Q(n)$, then $\Sigma Q(d)G(\delta) = \Sigma P(d)H(\delta)$; the summations for all values d, δ such that $d\delta = n$." The proof of this is immediate by the foregoing method; also the general case of elimination between m such equations is easily derived in compact form. Applying these results to the functions found in the first part, an indefinite number of relations are derived.

5. The results for the number of representations of an integer m as the sum of n squares have been completed by Glaisher for $n \not> 18$, in a form which involves only functions of n, including a particular function dependent upon the number of primary numbers having m as norm. In Dr. Bell's second paper the aim is different, and it is required to express the numbers of representations in terms of real divisors of numbers not exceeding the number to be represented, and in terms of the numbers of representations of special numbers in the given form. As usual, Euler's second method is used, viz., logarithmic differentiation of theta identities. From these, if originally s theta series and products are multiplied together, are found recurrence relations connecting the numbers of representations of m as the sum of s squares of specified linear forms, e. g., either all odd or s' odd and $s - s'$ even, and numerical functions of the numbers 1, 2, $\cdots$, $m - 1$.

In the cases where these recurrence relations hold between the functions for the numbers themselves, and not their squares, they may be used to furnish a sufficient set of linear equations from which the numbers of representations are deduced. If the recurrences are between functions of squares, the resulting set of equations is in general deficient, and cannot be used. From this standpoint the functions $X_i(n)$, $\Theta(n)$, $W(n)$ of Glaisher are replaced by their equivalents of the form $\Sigma f(n)g(n-r)$, $r = 0, \cdots, n$, where $f(n)$, $g(n)$ are numerical functions depending on n alone. The representations as far as 24 squares are easily found thus. As an example of the nature of the formulæ, if $\psi_s(s+8n)$ is the total number of decompositions of $s+8n$ into the sum of s odd squares, then

$$\psi_s(8n+s) = \frac{1}{n!}(-1)^n s^{n-1} D,$$

where

$$D = \begin{vmatrix} \Delta(n)+\Delta(n-1)\cdot\psi_s(s+8) & 0 & \Delta(2) & \Delta(3) & \cdots & \Delta(n-2) \\ \Delta(n-1)+\Delta(n-2)\cdot\psi_s(s+8) & -\frac{n-1}{s} & 0 & \Delta(2) & \cdots & \Delta(n-3) \\ \Delta(n-2)+\Delta(n-3)\cdot\psi_s(s+8) & 0 & -\frac{n-2}{s} & 0 & \cdots & \Delta(n-4) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta(3)+\Delta(2)\cdot\psi_s(s+8) & 0 & 0 & 0 & \cdots & \Delta(2) \\ \Delta(2) & 0 & 0 & 0 & \cdots & -\frac{2}{s} \end{vmatrix}$$

and $\Delta(m)$ = the excess of the sum of the odd divisors of m over the sum of the even divisors of m.

6. Mr. Clements presents the following results, supplementary to those published by him in the June number of the BULLETIN. If the transformation

$$T: \quad x = f(u, v), \quad y = \varphi(u, v),$$

where $f(u, v)$ and $\varphi(u, v)$ are single-valued and analytic functions of the complex variables u and v in the point (0, 0) and vanish there, is defined throughout a complete neighborhood R of $(u, v) = (0, 0)$, it defines a neighborhood $\overline{R}$ of the point $(x, y) = (0, 0)$. If to some point (x, y) of the region $\overline{R}$ there correspond from T, m points (u, v) of R, and if to no point of $\overline{R}$ do there correspond more than m points (u, v) of R, then there exists an m-valued inverse defined throughout the complete neighborhood of $(x, y) = (0, 0)$, everywhere continuous, analytic except along a complex one-dimensional locus where it is less than m-valued, and having the value $u = 0$, $v = 0$ when $x = 0$, $y = 0$.

Let

$$J_1 = \frac{D(f, \varphi)}{D(u, v)}, \quad J_{m-1} = \frac{D(f, J_m)}{D(u, v)}.$$

If $J_1(0, 0) = \cdots = J_{n-1}(0, 0) = 0$, $J_n(0, 0) \neq 0$, and if in the point (0, 0) J_{n-1} is a factor of every J_m with smaller subscript, then T is equivalent to transformations one-to-one and analytic both ways, combined with a single transformation of the form $x = u$, $y = v^n$.

7. Professor Kasner shows that the only element transformations which convert integrable equations of the form $Adx + Bdy + Cdz = 0$ into integrable equations are the contact transformations.

8. The theorem proved by Dr. Taylor is the following:

Let $f(z)$ be a function which is single-valued and analytic throughout the interior of a region S of the z-plane. If $f(z)$ vanishes at every point of a connected portion of the boundary, two points of which can be joined by a curve C lying wholly within S, then $f(z) \equiv 0$.

A proof of this theorem was given by Painlevé, *Toulouse Annales*, volume 2 (1888), page B. 29, for the case where the portion of the boundary along which $f(z)$ vanishes is an arc of a regular curve. The proof given in the present paper holds for the general case for which the theorem is stated.

9. The character of a transformation of n variables (real or complex) in the neighborhood of a point where its Jacobian

determinant does not vanish is well known. The purpose of Dr. Dederick's paper is to show that, in general, the transformation of the neighborhood of a point where $J = 0$ is essentially similar to the transformation $y_1 = x_1^2$, $y_i = x_i$, $(i = 2, \cdots, n)$ in the neighborhood of a point where $x_1 = 0$. This is done by breaking up the given transformation into three successive transformations, of which the first and last have non-vanishing Jacobians and the second is of the form indicated. Apart from considerations of continuity the only restriction on the application of this process is the requirement of the non-vanishing of at least one of the determinants obtained from J by replacing one of the functions in it by J itself.

10. Professor Volterra considers his integral equation of the first kind under a new form which has a special significance in the theory of the composition of the first kind. He considers the fundamental problem of finding all the functions which are permutable with a given function, and shows that these problems are only particular cases of a new type of linear integral equations. He studies the equations of this type and more especially the equation

$$(1) \qquad \int_x^y \varphi(x, \xi) f(\xi, y) d\xi + \int_x^y f(x, \xi) \psi(\xi, y) d\xi = \theta(x, y),$$

where φ, ψ, θ are given functions and f is the unknown function.

The cases are distinguished: where this equation has an infinite number of finite solutions, where there is only a single finite solution, and where there is no solution.

In this connection the author demonstrates the general theorem: The necessary and sufficient condition that the equation

$$(2) \qquad \int_x^y \varphi(x, \xi) f(\xi, y) + \int_x^y f(x, \xi) \psi(\xi, y) d\xi = 0$$

has an infinite number of solutions, φ, ψ being given functions of the first order and f the unknown function of the first order, is that

$$(3) \qquad \varphi(x, x) + \psi(x, x) = 0.$$

To prove this theorem it is necessary to recall that a function $f(x, y)$ is of the first order when $f(x, x) \leqq 0$. When (3) is

satisfied, all the solutions of equation (2) depend on an arbitrary function of one variable, and to obtain them it is necessary to solve a partial integro-differential equation of the first order. The solution is given by a series which is always convergent.

Similarly to solve the integral equation (1) it is necessary to emp'oy an integro-differential equation.

Finally, considering the relation

$$\int_x^y f(x, \xi)\varphi(\xi, y)d\xi = \psi(x, y),$$

the author employs the symbol $\overset{\bullet}{f}\overset{\bullet}{\varphi} = \psi$ and introduces the new symbols $f = \overset{\bullet}{\psi}\overset{\bullet}{\varphi}^{-1}$, $\varphi = \overset{\bullet}{f}^{-1}\overset{\bullet}{\psi}$.

The symbol $\overset{\bullet}{f}^{-1}$ leads to a theory with regard to functions analogous to the relation of fractions to the integers. The author gave an account of this theory.

11. The fact that to a properly discontinuous group of linear transformations of a complex variable there correspond single-valued analytic functions which are invariant under the transformations of the group was shown by Poincaré by means of the automorphic theta functions. It is conceivable, however, that the functions thus formed may admit the transformations of a larger group which contains the given group as a subgroup. The object of Professor Osgood's paper is to show that there are functions which admit the transformations of the given group, but of no larger group.

12. Professor Birkhoff presented a proof of the theorem of Poincaré recently enunciated in the *Rendiconti del Circolo Matematico di Palermo* (volume 33 (1912), pages 375–407). This proof will be published in the coming January number of the *Transactions* of the Society.

13. Professor Huntington's paper gives a new set of postulates for ordinary euclidean three-dimensional geometry. The postulates involve only two variables: (1) a symbol K, which may denote any class of elements $A, B, C, \cdots$; and (2) a symbol R, which may denote any relation ARB between two of these elements. The most familiar system (K, R) which satisfies all the postulates is the system in which K is the class

of all spheres of diameter not less than some constant $c(c \geqq 0)$, and R is the relation of inclusion, so that ARB means "A within B." Any two systems (K, R) which satisfy all the postulates are shown to be isomorphic with respect to the variables K and R; so that any such system is logically identical with the geometric system just mentioned. The postulates therefore form a "categorical set" by which the geometric type of system (K, R) is completely determined. The set contains eighteen "formal laws" which are shown to be independent of one another, and seven "existence postulates" which are shown to be independent of one another and of the formal laws. The most important definitions are the following: A sphere is any element of the class K. A point is a sphere which contains no other sphere within it. If A and B are two points, the segment $[AB]$ is the class of points X such that any sphere which contains A and B will also contain X. If A, B, C are three points, the triangle $[ABC]$ is the class of points X such that every sphere which contains A, B, and C will also contain X. The line AB is the class of points belonging to the segment $[AB]$ or to either of its two prolongations, $[AB']$ and $[BA']$. Here $[AB']$, for example, is the class of points X such that $[XB]$ contains A. The plane ABC is the class of points belonging to the triangle $[ABC]$ or to any of its six extensions. Here the vertical extension $[AB'C']$, for example, is the class of points X such that $[BCX]$ contains A; and the lateral extension $[ABC']$, for example, is the class of points X such that $[AB]$ and $[CX]$ have a common point. Two lines are parallel if they belong to the same plane and have no point in common. The mid-point of a segment $[AB]$ is the (unique) point of intersection of the diagonals of a parallelogram constructed on $[AB]$ as one diagonal. The center of a sphere is a (unique) point O within the sphere such that every pair of chords containing O are the diagonals of a parallelogram. Here a chord of a sphere is a segment whose end points are within the sphere while both its prolongations are outside. By the aid of the definitions of mid-point of a segment (which gives translation) and the center of a sphere (which gives rotation) it is then easy to define the congruence of two segments. All these definitions, it should be noticed, are in terms of the fundamental variables K and R. The paper will be published (in English) in the *Mathematische Annalen*.

14. The paper of Dr. Jackson is primarily a study of the degree of convergence of the series obtained by integrating or differentiating a given Fourier series a given number of times. The Abelian device of partial summation suffices if the number of integrations or differentiations is even; if this number is odd, recourse is had to a theorem communicated by the author at a recent meeting of the Society, concerning the approximate representation of an indefinite integral by a finite trigonometric sum. The following results are of a more special nature:

If $f(x)$, a function of period 2π, has a $(k-1)$th derivative satisfying a Lipschitz condition with coefficient λ, then $f(x)$ is represented by the partial sum of its Fourier's series to terms of the nth order ($n \geqq 5$), with an error not exceeding $36\lambda \log n/n^k$. If k is odd, the coefficient 36 may be replaced by 12.

If the Fourier series $\Sigma(a_n \cos nx + b_n \sin nx)$ converges uniformly so that the remainder after terms of the nth order does not exceed $\varphi(n)$, where $\Sigma(\varphi(n) \log n)/n$ converges and $\lim_{n=\infty} \varphi(n) . \log n = 0$, then the series $\Sigma(a_n \sin nx - b_n \cos nx)$ converges uniformly.

15. In this paper, Mr. Bennett points out that a solution in positive numbers of a system of linear algebraic equations with positive coefficients is possible when and only when the given equations can be reduced to a certain normal form. The proof depends immediately upon two simple lemmas of n-dimensional geometry. For a large class of cases sufficient conditions that the positive quantities occurring in a so ution shall be integers, are obtained by elementary geometrical methods.

F. N. COLE,
Secretary.

THE TWENTY-SECOND REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-second regular meeting of the San Francisco Section of the Society was held at the University of California on October 26, 1912. About twenty persons were present, including the following members of the Society:

Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor L. M. Hoskins, Dr. Frank Irwin, Dr. C. G. P. Kuschke, Professor D. N. Lehmer, Professor J. H. McDonald, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Professor E. W. Ponzer, Professor T. M. Putnam and Professor A. W. Whitney.

One session was held beginning at 10:30 A.M., Professor Hoskins, chairman of the section, presiding. The following officers were elected for the ensuing year: chairman, Professor Edwards; secretary, Professor Manning; program committee, Professors Manning, Noble, and Moreno.

It was voted to hold the next meeting at Stanford University on April 12, 1913. The members present lunched together after the meeting at the Faculty Club.

The following papers were presented at this meeting:

(1) Professor D. N. LEHMER: "On the expansion of the pure surd $R^{1/\lambda}$."

(2) Professor A. W. WHITNEY: The representation upon a tetrahedron of the logical relations of two classes."

(3) Professor C. I. LEWIS: "A new algebra of implications." Professor Lewis was introduced by Professor Whitney.

Abstracts of the papers follow below.

1. The nth complete quotient of the surd $R^{1/\lambda}$ being written in the form

$$\frac{A_n R^{\frac{\lambda-1}{\lambda}} + B_n R^{\frac{\lambda-2}{\lambda}} + C_n R^{\frac{\lambda-3}{\lambda}} + \cdots + K_n R^{\frac{1}{\lambda}} + P_n}{Q_n},$$

Professor Lehmer shows that the coefficients A_n, B_n, C_n, $\cdots$, K_n, P_n, and Q_n are expressible as follows:

$$A_n = \gamma^{\lambda-2},\quad B_n = \alpha\gamma^{\lambda-3},\quad C_n = \alpha^2\gamma^{\lambda-4},\quad \cdots,\quad K_n = \alpha^{\lambda-2},$$
$$(-1)^{n-1}Q_n = \alpha^\lambda - R\gamma^\lambda,\ (-1)^n P_n = \alpha^{\lambda-1}\beta - R\gamma^{\lambda-1}\delta,$$

where α/γ and β/δ are the nth and $(n-1)$th convergents in the expansion of $R^{1/\lambda}$. Other remarkable relations between the coefficients are indicated.

2. From two logical classes can be developed a system of sixteen elements which is closed under the operations of logical addition, multiplication, and negation and forms a group under a certain other operation. Professor Whitney discusses this system and shows that it has a high degree of symmetry and that its internal structure is the same as that of a regular tetrahedron.

3. In the paper of Professor Lewis there is set up a revised system of implications in the algebra of logic which will exclude such doubtful theorems as "A false proposition implies any proposition" and "A true proposition is implied by any proposition." This system indicates that definitions in mathematics are relations of reciprocal implication, and that such relations can sometimes be deduced instead of assumed.

T. M. PUTNAM,
Secretary of theSection.

THE FIFTH INTERNATIONAL CONGRESS OF MATHEMATICIANS. SECTIONS II–IV.

SECTION II. GEOMETRY.

IN geometry, four sectional meetings were held, the chairmen being H. F. Baker, F. Severi, J. Drach and F. Morley. A. L. Dixon and E. Bompiani were elected permanent secretaries for all the sessions. The following papers were presented before this Section.

(1) BROUWER, L. E. J., Amsterdam: "Sur la notion de classe de transformations d'une multiplicité."

(2) MORLEY, F., Baltimore: "On the extension of a theorem due to W. Stahl."

(3) EISENHART, L. P., Princeton: "Continuous deformation of surfaces applicable to quadrics."

(4) BOMPIANI, E., Rome: "Recent progress in projective differential geometry."

(5) NEVILLE, E. H., Cambridge, England: "On generalized moving axes."

(6) BRÜCKNER, M., Bautzen: "Ueber Raumteilung durch 6 Ebenen und die Sechsfläche."

(7) MARTIN, A., Washington: "On rational right-angled triangles."

(8) STÉPHANOS C., Athens: "Sur l'équivalent analytique du problème des principes de la géométrie."

(9) ESSON, W., Oxford: "On the characters of plane curves."

(10) DRACH, J., Toulouse: "Résumé de recherches géométriques."

(11) GROSSMANN, M., Zürich: "Die Zentralprojection in der absoluten Geometrie."

(12) SCHOUTE, P. H., Groningen: "On the characteristic numbers of the polytopes $e_1e_2 \cdots e_{n-1}S_{n+1}$ and $e_1e_2 \cdots e_{n-1}M_n$ of space S."

(13) KASNER, E., New York: "Conformal geometry."

(14) TZITZEICA, G., Bucharest: "Sur les surfaces isothermiques."

(15) SOMMERVILLE, D. M. Y., St. Andrews: "The pedal line of the triangle in non-euclidean geometry."

(16) HOSTINSKÝ, B., Prague: "Sur les Hessiennes successives d'une courbe du troisième degré."

(17) FINSTERBUSCH, J., Zwickau: "Geometrische Maxima und Minima mit Anwendung auf die Optik."

(18) HUDSON, Miss H. P., Croydon: "On binodes and double curves."

(19) STUDY, E., Bonn: "Conformal mapping of complex domains."

(20) HATZIDAKIS, N., Athens: "Sur les paires de trièdres de Frenet."

(21) KÖNIG, D., Budapest: "Zur analysis situs der Doppelmannigfaltigkeiten und der projectiven Räume."

(22) SINZOV, D., Charkow: "Sur la théorie des connexes."

(23) JANISZEWSKI, Z., Warsaw: "Ueber die Begriffe Linie und Fläche."

(24) WEITZENBÖCK, R., Bonn: "Ueber das sechs-Ebenen-Problem im R_4."

1. Two uniform and continuous representations, one of a closed multiplicity μ and the other μ', are said to belong to

the same class if it is possible to go from one to the other by continuous modification. Professor Brouwer proved that in the case in which both entities are spheres, all the representations of the same degree belong to the same class.

2. In the article of Stahl (*Crelle*, volume 104) referred to by Professor Morley, the following theorems appeared: Any conic on the double lines of a rational quartic determines by the other four common lines the fundamental involution. In the present paper the following theorems are added: any pencil of curves of class four on the double lines of a rational quintic contains five which touch the quintic. The five parameters of contact are in the fundamental involution.

Any set of curves of class six on the double lines of a rational sextic contains six which osculate the sextic. The six parameters of osculation are in the fundamental involution. Similarly for class n.

3. Professor Eisenhart's paper is in abstract as follows:

If S is a surface applicable to a quadric Q and S_1 is a Bianchi transform of S, the joins of corresponding points on S and S_1 form a W-congruence for which these surfaces are the focal surfaces. By the general theory of W-congruences one has accordingly an infinitesimal deformation of S and S_1. Professor Eisenhart has made use of these infinitesimal deformations to build a suite of continuous deforms of Q. Such a suite, called a *system* (Q), is determined by a set of five differential equations in two dependent and three independent variables, which can be shown to admit of analytic solutions. When a solution is known, the intrinsic functions of the surfaces are given directly. Systems (Q) occur in pairs, corresponding surfaces of the two systems being focal surfaces of a W-congruence. These systems admit transformations into systems of the same kind which are a generalization of the Bianchi transformations of a single surface. There exist systems (Q) of ruled surfaces applicable to Q. When in particular Q is an hyperboloid of revolution of one sheet, there arises incidentally a continuous deformation of Bertrand curves into curves of the same kind.

4. Dr. Bompiani summarized his report as follows:

The first results of projective differential geometry of

hyperspace appear in papers by Del Pezzo (1886) on spaces tangent to a given variety immersed in a larger one, and by Segre (1888) on systems of ∞^{n-1} lines in S_n.

In his more recent memoirs, published between 1896 and 1910, Professor Segre has treated this branch of geometry in a different way, which has opened up a new field. If we consider the curves of a variety V_m issuing from one of its points with a determinate osculating $S_\nu (\nu \geqq 0)$, the spaces S_n $(n > \nu)$ which osculate it belong to a linear space which is called the n-osculant to the given variety along the fixed S_ν (when it does not coincide with the former). The first problem (of local character) concerns the distribution of the osculating S_n among the n-osculants, the law of variation of these spaces with the S_ν, etc. A second problem (of general character) concerns the possibility of associating with V_m varieties of fewer dimensions to which the osculating spaces are related. This problem corresponds to that of the determination of a double system of conjugate lines on a given surface; it is stated in terms of one or more differential equations. An application to the study of ruled surfaces and certain curves upon them has been made by Wilczynski and others. In the theory of multiply infinite systems of lines, the analogous problem to that in S_3 of finding the developables of a given congruence is that of finding the ruled surfaces in the system and of finding the minimum indices of the various developables.

6. Dr. Brückner made a further study of the problem which he discussed at the Rome congress. He considered the depiction of the 42 complete figures composed of six planes and explained the method of construction. A number of the figures were illustrated by means of models.

9. In Professor Esson's paper the changes in the Plücker numbers were derived which take place when a curve of a pencil has a double point or cusp, and a criterion was deduced for discriminating between proper tangents and lines passing through double points or cusps. The dual cases were then considered and the corresponding formulas derived.

12. After having indicated the meaning of the expansion symbols $e_1, e_2, \cdots, e_{n-1}$ representing operations to be applied to the edges, faces, limiting bodies . . . of one of the three

regular polytopes (simplex, measure polytope, cross polytope) in order to get the semiregular polytopes corresponding to the semiregular polyhedra of Archimedes, and having recalled that the application of all these operations one after another to the measure polytope and the cross polytope leads to the same result, Professor Schoute explains how the characteristic numbers, i. e., the numbers of vertices, edges, faces, etc., of the two polytopes of space S_{n+1} can be deduced from those of the two polytopes of space S_n by means of a rule nearly as simple as the generally known rule of the triangle of Pascal. In the case of the simplex we have

S_2	6	6	1			
S_3	24	36	14	1		
S_4	120	240	150	30	1	
S_5	720	1800	1560	540	62	1
. . . . .						

in the case of the measure polytope we get

S_2	8	8	1			
S_3	48	72	26	1		
S_4	384	768	464	80	1	
S_5	3840	9600	8160	2640	242	1
. . . . .						

In both cases the unit always represents the thing itself. Now each number in these tables is deduced from the one that is immediately above it and the one on the same horizontal line as the latter one place to the left. This may be shown for both cases by indicating how we pass from the plane S_2 to ordinary three dimensional space S_3. For the simplex we find

number of faces $= \mathbf{2}(1 + 6)$
number of edges $= \mathbf{3}(6 + 6)$
number of vertices $= \mathbf{4}.6$

and for the measure polytope

number of faces $= \mathbf{2}.1 + \mathbf{3}.8$
number of edges $= \mathbf{4}.8 + \mathbf{5}.8$
number of vertices $= \mathbf{6}.8$

In these two cases there is only a difference as to the multipliers represented in heavy type.

After having given the geometrical proof of these rules as far as the step leading up from S_2 to S_3 is concerned, the author

indicates how the two forms obtained in S_n may be represented by the coördinates of their vertices. Thus the symbol (3, 2, 1, 0) represents the vertices of the polyhedron with the characteristic numbers (24, 36, 14) deduced from the simplex in a certain kind of barycentric coordinates, whilst $[1 + 2\sqrt{2}, 1 + \sqrt{2}, 1]$ represents in the same way in ordinary cartesian coordinates the vertices of the polyhedron with the characteristic numbers (48, 72, 26). In the case of (3, 2, 1, 0) the round brackets mean that we have to take all the permutations of the values; in the case of $[1 + 2\sqrt{2}, 1 + \sqrt{2}, 1]$ the square brackets indicate that we have to take all the permutations of the values with all the possible combinations of the signs plus and minus. By the introduction of these symbols the analytical proof with respect to the step from n to $n + 1$ is nearly as simple as that from 2 to 3.

In the end it was indicated that the proved recurrent relations lead up to the following results by means of the method of induction. If $a_{p,\,n}$ represents the number of limits $(l)_p$ of p dimensions of a polytope in S_n, we have

$$\text{for } e_1e_2 \cdots e_{n-1}\, S(n+1)$$

$$\begin{aligned} a_{n-1,\,n} &= 2(2^n - 1) \\ a_{n-2,\,n} &= 3(3^n - 2.2^n + 1) \\ a_{n-3,\,n} &= 4(4^n - 3.3^n + 3.2^n - 1) \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

$$\text{for } e_1e_2 \cdots e_{n-1}\, M_n$$

$$\begin{aligned} a_{n-1,\,n} &= 3^n - 1 \\ a_{n-2,\,n} &= 5^n - 2.3^n + 1 \\ a_{n-3,\,n} &= 7^n - 3.5^n + 3.3^n - 1 \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

the laws of succession of which are sufficiently transparent.

It is almost unnecessary to add that the operations of expansion were introduced by Mrs. A. Boole Stott ("General deduction of semiregular polytopes and space fillings of regular ones," *Verhandelingen* of Amsterdam, volume 11, number 1), but it is necessary to say that the geometrical proof given by the author was also suggested by her.

13. The geometry of the plane based upon the infinite group consisting of all conformal transformations has been developed only in those directions which are suggested by the

theory of functions of a complex variable. Certain simple and fundamental problems have therefore been overlooked. The simplest configuration of real interest is the curvilinear angle (two analytic arcs having a point in common). The conformal transformation is required to be regular all around the vertex. When are two angles equivalent? Equality of magnitude is of course necessary, but not always sufficient. Professor Kasner studies the differential invariants of higher order which thus arise. It makes an essential difference whether the magnitude of the angle is commensurable or incommensurable with respect to 180°. The case of the horn angle (two curves touching each other) is of special interest.

14. After having recalled the methods of representation which he had previously developed, Professor Tzitzeica applied them to a system of partial differential equations of the third and of the fourth order. In the latter case he obtained the equation of the fourth order derived by another method by Rothe and by Calapso.

15. The loci discussed by Professor Sommerville are defined as follows: In non-euclidean geometry, if the feet of the perpendiculars X, Y, Z from P upon the sides of a triangle ABC are collinear, the locus of P (pedal locus) is a bi-partite cubic passing through A, B, C; the envelope of XYZ (pedal envelope) is a curve of the third class touching the sides of the triangle. In euclidean geometry the pedal locus degenerates into the circum-circle and the line at infinity. The only other case in which the locus degenerates into a straight line and a circle is when the triangle is equilateral with imaginary angles $\cos^{-1} 2$.

16. Let m be the parameter in the equation of a plane cubic curve reduced to the Hessian canonical form, and r the absolute invariant. Professor Hostinsky considered the successive Hessians $H_1, H_2, \cdots$ of the given curve, regarded as functions of m, and in particular the condition for periodicity, $H_n = H_0$, in which H_0 is a given cubic. If $n = 1$, the syzygetic pencil obtained by varying m contains four curves which degenerate into three straight lines, each of which is identical with its Hessian.

For $n = 2$, there are three cycles composed of harmonic curves ($r = 0$). When $n = 3$, the condition is $r^2 + 3r + 3 = 0$,

and there are eight cycles. For every value of n the equation in r is always abelian. If H_k is bipartite, H_{k+1} and H_{k-1} are unipartite.

18. In the paper of Miss Hudson two theorems were discussed:

If a point O is an isolated binode which reduces the class of a surface F by i, then $F \equiv H \cdot K + f_i$, in which f_i is small of order i near O, and H, K are surfaces each passing once only through O, and having contact of order $i - 2$ with one sheet of F.

If F is of degree n and has t triple points lying upon a general nodal curve C of degree m and rank r with d nodes, then, if there are no other singularities, the number of pinch points on F is $2\{m(n-2) - r - 2d - t\}$, and the reduction in the class of the surface is $m(7n - 12) - 4r - 8d + 3t$.

20. The paper of Professor Hatzidakis dealt with the relations existing between the curvature of a pair of generalized trihedra of Frenet. The expressions for curvature and torsion of the one (D_1) are derived in terms of those of the other (D_2), and the direction cosines of (D_1, D_2) and also the condition that D_1 is a trihedron of Frenet with regard to D_2 are found. Finally, the relations between the normal and the geodesic curvatures, and also the geodesic torsion are derived.

21. As generalization of the concept of double surface, double varieties for hyperspace may be defined. In this way spheres appear as double varieties of projective spaces having the same number of dimensions. From these hypotheses, Dr. König derived the following theorem: The n-dimensional projective space is unilateral for even values of n, and bilateral for odd values.

23. Spatial intuition, by means of which we think we know something of the concepts curve and surface, often leads us astray. An example is furnished by those curves and surfaces having properties not compatible with intuition. Dr. Janiszewski discussed the two cases: first, a curve having a strip in common with every plane of a parallel pencil, and holomorphic with a plane curve whose multiple points compose a continuum; second, a surface which cannot contain any repre-

sentation of the area of a circle—namely, a cylindrical surface whose directrix is a plane curve without a simple arc.

24. When six planes are given in R_4, then in the general case there are five lines which these planes cut. Dr. Weitzenböck explained the geometric significance of this number and derived an equation of order five whose roots determine the five lines.

Section III*a*. Mechanics, Physical Mathematics, Astronomy.

The programme consisted of the following papers:

Turner, H. H., Oxford: "On double lines in periodograms."

Moulton, F. R., Chicago: "Relations of families of periodic orbits in the restricted problem of three bodies."

Föppl, L., Göttingen: "Stabile Anordnungen von Elektronen im Atom."

Smoluchowski, M. S., Lemberg: "On the practical applicability of Stokes's law of resistance and the modifications of it required in certain cases."

Love, A. E. H., Oxford: "The application of the method of W. Ritz to the theory of the tides."

Leuschner, A. V., Berkeley: "The Laplacian orbit methods."

Bennett, G. T., Cambridge: "The balancing of the four-crank engine."

Kármán, Th. von, Göttingen: "Luftwiderstand und Hydrodynamik."

Bromwich, T. J. I'a., Cambridge: "Some theorems relating to the resistance of compound conductors."

Ewald, P. P., Göttingen: "Dispersion and double refraction of electrons in rectangular grouping (crystals)."

Miller, D. C., Cleveland: "The graphical recording of sound waves; effect of free periods of the recording apparatus."

Terradas, E., Barcelona: "On the motion of a chain."

Abraham, M., Milan: "The gravitational field."

McLaren, S. B., Birmingham: "Aether, matter, and gravity."

Silberstein, L., Rome: "Self-contained electromagnetic vibrations of a sphere as a possible model of the atomic store of latent energy."

Somigliana, C., Torino: "Sopra un criterio di classificazione dei massimi e dei minimi delle funzioni di piu variabili."

ESSON, W., Oxford: "On a law of connection between two phenomena which influence one another."

BLASCHKE, W., Eldena: "Reziproke Kräftepläne zu den Spannungen in einer biegsamen Haut."

BLUMENTHAL, O., Aachen: "Ueber asymptotische Integration von Differentialgleichungen mit Anwendung auf die Berechnung von Spannungen in Kugelschalen."

BOULAD, F., Cairo: "Extension de la notion des valeurs critiques aux équations à 4 variables d'ordre nomographique supérieure."

BRODETSKY, S., Cambridge: "The solution of dynamical problems."

DENIZOT, A., Lemberg: "Theoretisches über den freien Fall eines Körpers bei rotierenden Erde."

DOUGALL, J., Kippen: "The method of transitory and permanent nodes in the theory of elasticity."

HAGEN, J. G., Rome: "How the Atwood machine proves the rotation of the earth, even quantitatively."

LAMB, H., Manchester: "On wave-trains due to a single impulse."

SAMPSON, R. A., Edinburgh: "Some points in the theory of errors."

SECTION III*b*. ECONOMICS, ACTUARIAL SCIENCE, STATISTICS.

The following papers were read:

LEHFELDT, R. A.: "Equilibrium and disturbance in the distribution of wealth."

AMOROSO, L.: "I caratteri matematici della scienza economica."

SHEPPARD, W. F., Surrey: "Reduction of errors by means of negligible differences."

BRODIE, R. R., Edinburgh: "Curves of certain functions relating to mortality and compound interest."

PEEK, J. H.: "Application of the calculus of probabilities in calculating the amount of securities, etc., in the Dutch State Insurance Office."

QUIQUET, A., Paris: "Sur une méthode d'interpolation exposé par Henri Poincaré et sur une application possible aux fonctions de survie d'ordre *n*."

STEFFENSEN, J. F. A. F., Hellerup: "On the fitting of Makeham's curve to mortality statistics."

EDGEWORTH, F. Y., Oxford: "A method of representing frequency groups by analytic geometry."

HERON, D.: "Fallacious methods of measuring association."

SHEPPARD, W. F., Surrey: "The calculation of moments of an abrupt frequency distribution."

ARANY, D., Budapest: "Ein Beitrag zur Laplace'schen Theorie der erzeugenden Funktion."

GÉRARDIN, A., Nancy: "Statistique des vingt séries parues du Répertoire Bibliographique des Sciences Mathématiques."

SECTION IV*a*. PHILOSOPHY, HISTORY.

The list of titles of papers is as follows:

BURALI FORTI, E.: "Sur les lois générales pour l'algorithme des symboles de fonction et d'opération."

BLUMBERG, H.: "Ueber ein Axiomen-System für die Arithmetik."

ENESTRÖM, G.: "Resolution relating to the publication of G. Valentin's general Bibliography of mathematics."

GÉRARDIN, A.: "Note historique sur la théorie des nombres."

HARDING, P. J.: "The geometry of Thales."

HUNTINGTON, E. V.: "A set of postulates for abstract geometry expressed in terms of the simple relation of inclusion."

ITELSON, G.: "Bemerkungen über das Wesen der Mathematik."

ITELSON, G.: "Thomas Solly von Cambridge als Logistiker."

JOURDAIN, P. E. B.: "Isoid relations and the modern theory of irrational numbers."

JOURDAIN, P. E. B.: "Fourier's influence on pure mathematics."

JOURDAIN, P. E. B.: "The ideas of the 'fonctions analytiques' in Lagrange's early work."

LORIA, G.: "Intorno ai metodi usati dagli antichi greci per estrarre le radici quadrate."

MUIRHEAD, R. F.: "Superposition as a basis for geometry; its logic and its relation to the doctrine of continuous quantity."

PADOA, A.: "Comparison entre la logique de l'extension et la logique de la compréhension."

PADOA, A.: "Une démonstration du principe d'induction complète."

RUDIO, F.: "Mitteilungen über die Eulerausgabe."

VACCA, G.: "Sul valore della ideografia nella espressione del pensiero; differenze caratteristiche tra ideografia e linguaggio ordinario."

VACCA, G.: "On some points in the history of the infinitesimal calculus; relations between English and Italian mathematicians."

ZERMELO, E.: "Ueber die Grundlagen der Mengenlehre."

ZERMELO, E.: "Ueber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels."

ZERMELO, E.: "Ueber axiomatische und genetische Methoden bei der Grundlegung mathematischer Disciplinen."

SECTION IV*b*. DIDACTICS.

Section IV*b* held six meetings (two of them jointly with Section IV*a*), under the chairmanship of the following gentlemen in order: Hon. B. A. W. Russell, C. Godfrey, David Eugene Smith, A. Gutzmer, E. Czuber, C. Bourlet, J. W. A. Young, Sir J. J. Thomson, R. Fujisawa, C. Godfrey.

Three of these sessions were held jointly with the International Commission for the Teaching of Mathematics. The first of these was opened by an address of welcome by the chairman, C. Godfrey. Thereupon, the following address on the work of the Commission was delivered by David Eugene Smith, who had been in recent conference with the president of the Commission, Professor Klein, and the central committee:

"As has already been mentioned, Professor Klein, to whose great energy and wisdom the success of the International Commission on the Teaching of Mathematics is largely due, is unable to be present, on account of illness. It was my privilege to propose to the delegates at our meeting on Wednesday the sending of a telegram to Professor Klein, and I now propose the same message to Section IV, as follows: 'The International Commission on the Teaching of Mathematics, and Section IV, at their first Cambridge meeting, express regret at your absence and best wishes for your recovery.' *

"The Commission was organized for the purpose of reporting upon the present status of the teaching of mathematics in the various countries of the world. Special sub-committees have also been appointed from time to time, to consider questions of international rather than merely national interest.

* By unanimous vote the telegram was duly sent to Professor Klein.

About one hundred and fifty reports on the work done in the various countries have been prepared, and at least fifty more are in contemplation. A world-wide interest in the improvement of mathematical teaching has been awakened, and the influence of the movement is certain to be very far reaching. Ten countries have completed the task set for themselves. In chronological order of completion these countries are Sweden, Holland, France, Switzerland, Austria, Japan, the United States of America, the British Isles, Hungary, and Denmark. In process of publication are the monumental work of Germany, with twenty-seven out of thirty-six reports already printed, and the reports of Italy, Roumania, Spain, and Russia. In contemplation are the reports of Greece, Norway, Australia, Portugal, Servia, and doubtless of several other countries.

"As to the future work of the Commission, the Central Committee earnestly desires that it be authorized to see to the completion of the reports. It is therefore very desirable that it be continued in power, both for this purpose and for the consideration of certain questions of great international significance. Such topics as the proper training of engineers, of calculus in the secondary schools, of the general value of intuition in the teaching of mathematics, of the training of teachers, and of the educational (cultural, disciplinary, non-technical) value of mathematics, may properly occupy the attention of the Commission in the next four years. Special conferences having already been held, at Bruxelles and Milano; it is proposed, if the committee is continued in power, to hold others between now and the time of the meeting of the Congress of 1916, if that shall be the date. Possibly such conferences may be held in France in 1914, in Germany in 1915, and in Stockholm in 1916.

"It is also hoped that each country will prepare a summary of the large features of the reports of other countries, to the end that the work that has been accomplished may have its full effect. It is further hoped that the various countries will continue the financial support that has been given to the central committee in the past.

"A word should be said at this time in memory of those distinguished teachers who have been connected with the movement, but who have been called from their labors to solve the great problem. Soon after the last Congress adjourned,

Professor Vailati, of Rome, a distinguished writer and an accomplished scholar, passed away. Scarcely in his full prime of life, his loss is felt not by Italy alone, but by all who appreciate scholarship and high educational standards. Professor Bovey, president of the Imperial Technical College at South Kensington, who was charged with the labor of reporting for Canada, has also been called from us; in his death the world lost a scholar and an administrator of prominence. And as he was planning to attend this Congress, four weeks ago to-day, Geheimrath Professor P. Treutlein of Carlsruhe, passed suddenly away. In his death Germany lost one of her foremost educators, and the International Commission one of its best supporters.

"We shall now proceed to the election of the officers for the next session, and then to the reception of the reports. The central committee has consulted with the committee on organization and it has been decided that the first set of reports shall be presented to the library of the University of Cambridge, a second set to our official hosts, the Cambridge Philosophical Society, and a third set to that great world-library, the library of the British Museum."

The general secretary of the Commission next made a statement as to the work of the central committee, and submitted its collected publications.

Thereupon the reports of the various countries were formally submitted to the congress. The countries were called in alphabetical order in the French language, and the following members of the Commission presented the reports, with a brief oral description, and a longer written statement, which will be published in the Proceedings of the Congress, and in the official organ of the Commission, *L'Enseignement Mathématique*. The statement accompanying the American report will be published in *School Sciences and Mathematics*, the official American organ.

Germany, Professor A. Gutzmer (Halle); Austria, Professor E. Czuber (Vienna); Belgium, Principal E. Clevers (Ghent); Denmark, Professor H. Fehr; Spain, Professor Toledo (Madrid); United States, Professor J. W. A. Young (Chicago); France, Professor C. Bourlet (Paris); Greece, Professor H. Fehr; Holland, Professor J. Cardinaal (Delft); Hungary, Professor E. Beke (Buda-Pesth); British Isles, Professor C. S. Jackson (Woolwich); Italy, Professor G. Castelnuovo (Rome); Japan, Pro-

fessor R. Fujisawa (Tokio); Norway, Professor M. Alfsen (Christiania); Portugal, Professor F. J. Teixeira (Oporto); Roumania, Professor G. Tzitzeica (Bucharest); Russia, Professor H. Fehr; Sweden, Professor H. Fehr; Switzerland, Professor H. Fehr (Geneva).

Also the following associated countries: Brazil, Professor E. de B. R. Gabaglia (Rio de Janeiro); Servia, Professor M. Petrovitch (Belgrade).

At the second joint session of Section IV*b* and the International Commission, the report of sub-commission *B*, on "The mathematical education of the physicist in the university," was presented by Professor C. Runge, and followed by a lively discussion.

At the last joint session of Section IV*b* and the International Commission, C. Goldziher presented a report on the work done by David Eugene Smith and himself towards preparing a bibliography of works on the teaching of mathematics, published since 1900. (This bibliography is about to be published by the United States Bureau of Education and can be obtained from the Bureau on request.) Upon motion of Professor Smith the following resolution was passed:

Resolved: that Section IV*b* of the International Congress of Mathematicians, assembled at Cambridge, expresses its thanks to the Honorable the United States Commissioner of Education for his great interest in publishing, for free distribution, the recent bibliography on the teaching of mathematics (1900–1912), and the hope that it may, through his good offices, be brought to completion to the year 1915, with such additions to the present list as may seem desirable.

David Eugene Smith then presented the report of Subcommission *A* on: "Intuition and experiment in mathematical teaching in secondary schools." The presentation of the report was followed by an extended discussion. This report will be published in the various official organs named above.

In the other sessions of Section IV*b* the following papers were presented:

WHITEHEAD, A. N.: "The principles of mathematics in relation to elementary teaching."

SUPPANTSCHITCH, R.: "Le raisonnement logique dans l'enseignement mathématique universitaire et secondaire."

HILL, M. J. M.: "The teaching of the theory of proportion."

HATZIDAKIS, N.: "Systematische Rekreationsmathematik in den mittleren Schulen."

GÉRARDIN, A.: "Sur quelques nouvelles machines algébriques."

CARSON, G. ST. L.: "The place of deduction in elementary mechanics."

NUNN, T. P. "The proper scope and method of instruction in the calculus in schools."

It was not possible to secure brief abstracts of the above papers for incorporation in this report. The papers will be published in full in the Proceedings of the Congress, and elsewhere.

Sir G. Greenhill made the following statement in regard to the work of the International Commission on the teaching of mathematics.

"The statement I have to make, Sir, to the Congress, is given in the formal words following:

1. The International Commission on the Teaching of Mathematics was appointed at the Rome Congress, on the recommendation of the members of Section IV.

2. The several countries, in one way or another, have recognized officially the work, and have contributed financial support.

3. About 150 reports have been published, and about 50 more will appear later.

4. The Commission will report in certain sessions of Section IV.

5. The Commission hopes to be continued in power, in order that the work now in progress may be brought to completion.

A resolution to this effect will be offered at the final meeting of the Congress."

The third general session closed the Congress. At this session, upon motion of C. Godfrey, seconded by W. von Dyck, the following resolution was unanimously passed. Its adoption had previously, upon the motion of Sir George Greenhill, been unanimously recommended to the Congress by the International Commission on the Teaching of Mathematics, in its separate session, and by Section IV*b* in joint session with the Commission.

Resolved: that the Congress expresses its appreciation of the support given to its Commission on the Teaching of Mathematics by various governments, institutions, and individuals; that the Central Committee composed of F. Klein

(Göttingen), Sir G. Greenhill (London) and H. Fehr (Geneva) be continued in power and that, at its request, David Eugene Smith, of New York, be added to its number; that the delegates be requested to continue their good offices in securing the cooperation of their respective governments, and in carrying on the work; and that the Commission be requested to make such further report at the Sixth International Congress, and to hold such conferences in the meantime as the circumstances warrant.

VIRGIL SNYDER.

THE MÜNSTER MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held in affiliation with the eighty-second convention of the association of German naturalists and physicians at Münster in Westphalia, September 15 to 19, under the presidency of Professor W. v. Dyck.

As usual, ample provision was made for the entertainment of the guests, a number of excursions being arranged for the afternoons, and a reception or concert each evening. Perhaps the most interesting excursion was that to the neighboring city of Essen, to inspect the works of the Krupp manufacturing company. The guests were shown the manifold processes of casting and hardening steel, the preparation of armor, the boring of cannon, etc., and were also furnished an opportunity of seeing the domestic and social problems and usages connected with this gigantic enterprise.

The session of Tuesday afternoon was devoted to the administrative affairs of the society. Reports of the status of the Encyclopedia and of the International commission on mathematical instruction were read, as well as a statement concerning the publication of various other works in which the society is interested, primarily those of Schroeder and of Euler. At this session Professor W. KILLING read a paper "On the preparation of the gymnasium teacher," which was followed by a general discussion.

The session of Wednesday morning was held jointly with the section of physics, upon the invitation of the latter, to listen to the following more general reports in mathematical physics:

(1) D. HILBERT (Göttingen): "Begründung der elementaren Strahlungstheorie."
(2) W. NERNST (Berlin): "Ueber den Energiegehalt der Gase."
(3) V. SMOLUCHOWSKI (Lemberg): "Experimentell nachweisbare, der üblichen Thermodynamik widersprechende Molekularphänomene."

Professor Smoluchowski presented a systematic resumé of certain phenomena, obtained experimentally, based upon a general formula of statistical mechanics derived from the fact that molecular systems execute small automatic oscillations from the state of thermodynamic equilibrium, according to the law of chance. Here belong in particular the Brownian molecular motion of particles suspended in liquids or gases, the distribution of emulsion particles in a gravitational field, the irregularity of the distribution of particles in solutions of collodion investigated by Svedberg, the appearance of opalescence in gases and mixtures in states approximating the critical ones, the blueness of the sky, certain manifestations of electric and optical phenomena observed in emulsions and in suspended particles, etc. The author called attention to certain other important phenomena, theoretically expected, which await experimental confirmation, and mentioned the width of the lines of the spectrum of Fabry, which, according to the Doppler principle, furnish an excellent justification of the Maxwell law of velocity. All these facts decide the long conflict between thermodynamics and kinetic theory in favor of the latter, so that it follows that the second fundamental law of thermodynamics, as formulated by Clausius, Thomson, and others is not true, since in small intervals of time and of space contradictory processes are continually going on, which are also observable in physical phenomena. The law is still applicable to unlimited fields of time and space.

Three sectional meetings were held to listen to papers presented to the society itself, the titles of which follow. On account of the Cambridge meeting of the Fifth international congress of mathematicians, the programme was rather shorter and the attendance slightly smaller than usual, but both were sufficient to make the Münster meeting an important one in the history of the society.

It has been the custom during a number of years to pay particular attention to some one branch of mathematics at these meetings, and the proceedings have taken the form of symposiums and reports rather than the presentation of finished papers containing results of original research. While papers on other subjects were welcomed, the emphasis this year was primarily on differential geometry, and over half the papers treated of some phase of this subject.

(1) W. v. DYCK (Munich): "Ueber die singulären Stellen der Differentialgleichungen erster Ordnung zweiten Grades."

(2) E. H. MOORE (Chicago): "Remarks concerning relatively uniform sequences and series of functions."

(3) R. ROTHE (Clausthal): "Anwendung der Vektoranalysis auf Differentialgeometrie."

(4) H. WIENER (Darmstadt): "Ueber eine geometrische Theorie der algebraischen Formen."

(5) F. MEYER (Königsberg): "Ueber einen verallgemeinerten Krümmungsbegriff."

(6) E. SALKOWSKI (Charlottenburg): "Ueber die verschiedenen Begründungsarten der Differentialgeometrie."

(7) R. v. LILIENTHAL (Münster): "Ueber die Bestimmung der berührenden Kurve und Fläche bei Kurven- und Flächenscharen."

(8) W. VELTEN (Kreuznach): "Ueber die Funktionen, die aus der Jacobischen Ω-Funktionen entspringen."

(9) A. VOIGT (Frankfurt): "Mathematische Theorie des Tarifwesens."

(10) H. MOHRMANN (Carlsruhe): "Ueber beständig hyperbolisch gekrümmte Kurvenstücke."

Abstracts of a number of the papers follow; the numbers correspond to those of the titles above.

3. Professor Rothe maintains that vector analysis in differential geometry is in fact of far greater importance than furnishing simply an abbreviation in the proofs and a convenient formulation of results. In the earlier investigations, originated by Grassmann, Hamilton, Möbius, the main question was the translation of formulas already known into the language of vectors; in this process many abridgements were found possible in the presentation and proofs. If we confine ourselves to vector analysis in the narrower sense (algebra) no simpler processes appear than those employed

by Gauss, but when differential processes (analysis) are considered, it becomes a different matter. There we deal particularly with the gradient, divergence, and curl. By giving these concepts their proper physical meaning, and combining with them the usual elementary concepts of differential geometry, a great advance is possible. With every point in space a vector (usually one-valued) is associated, whose direction determines the lines of flow, and whose length determines the intensity of the field. Such considerations have been employed (among others) by Burali-Forti. He takes, for example, in the theory of surfaces as field vector the unit vector of the properly directed surface normal, and the lines of flow are determined as the lines of a normal congruence. This assumption materially simplified the formulas of the theory of surfaces.

After explaining the concept of a general field, a number of new results in the theory of surfaces were derived.

4. The theory of invariants of algebraic forms of order ν in one or more ρ-dimensional variables is contained within the theory of the forms which are linearly constructed from ν series $X, Y, \cdots, W$. The form $f(X, Y, \cdots, W) = a_1x_1 + \cdots + a_\rho x_\rho$ is a linear form of order ν in a space of ρ (homogeneous) dimensions R_ρ when its coefficients are linear forms of order $\nu - 1$ in the series $Y, \cdots, W$. Equated to zero it defines a *geometric correspondence*, which associates $\nu - 1$ arbitrarily chosen points $Y, \cdots, W$ as locus of the point X with one element of $\rho - 1$ dimensions (point, line, plane, . . .) in space R_ρ. If by interchanging the series of f new forms $f_1, f_2, \cdots f_{\tau-1}$ appear, then $\tau p = f + f_1 + \cdots + f_{\tau-1}$ is a polar form, that is, in it any two rows may be interchanged. Moreover, the form f is also associated with a null form $n = f - p$, that is, with a form which vanishes when the coordinates of any arbitrary point are substituted in all the series; it can always be written in such a manner that every term contains a factor $(x_iy_k - x_ky_i)$. The Clebsch-Gordan development in series divides a form f into its polar form and null form, and further divides the latter into forms whose terms contain $1, 2, 3, \cdots$ factors of the designated kind, while the associated remaining factor is a polar form. At the Dresden meeting of the Vereinigung, Professor Wiener explained the process of transvection as applied to binary forms, and thus reduced the

determination of the invariants of any form to the problem of determining the associated polar form of any given linear form. In the extension to forms in ρ variables (so far as completed) he employs, as method of construction, only the geometric construction of the linear forms and their linear systems, and as means of proof the permutation of the series and the Galois groups generated by it. In this manner, the application of any symbol in the new theory that does not have a geometric meaning is rejected.

6. In the paper of Professor Salkowski, the various directions of propagation along a curve were considered and the various concepts of curvature systematized. In particular, the value and significance of various related ideas, like the synthetic methods of Schell and of Mannheim, were criticized. A more extensive summary will appear in an early number of the *Jahresbericht*.

7. Leibnitz and l'Hospital were familiar with the fact that the various curves of a one-parameter family are all touched by one curve. Monge extended the idea to surfaces (Applications, § 6), and introduced the word envelope to define the surface thus generated, the word being at once suggested by visualization. If, for example, we take a cylinder whose cross-section extends to infinity on both sides of an inflexional tangent, and move the cylinder along the tangent, a family of surfaces results, each of which is touched by the same plane. Professor Lilienthal considered, in his Differentialgeometrie, Band I, the conditions under which the curves of a cross-section have as envelope the same section of the envelope of the surfaces. The same method is here immediately extended to surfaces.

Given a family of surfaces $f(x, y, z, v) = 0$, and a family of space curves

$$f = 0, \quad g(x, y, z, v) = 0; \tag{1}$$

or

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v); \tag{2}$$

the equations

$$\frac{\partial f}{\partial v} = 0, \quad \frac{\partial g}{\partial v} = 0 \tag{3}$$

are satisfied by (1). Conversely, if (1) and (3) satisfy $x = \varphi_1(v)$, $y = \varphi_2(v)$, $z = \varphi_3(v)$, the curve defined by these equations is a contact curve, provided the determinants $f_x g_z - f_z g_x$, etc., do not all vanish for $x = \varphi_1(v)$, etc. If they do vanish, the equations must be further examined. For the surfaces defined by (2) we consider the determinants

$$\frac{\partial f_2}{\partial u} \cdot \frac{\partial f_3}{\partial v} - \frac{\partial f_2}{\partial v} \cdot \frac{\partial f_3}{\partial u},$$

etc. If they do not all vanish we have a contact curve $u = \varphi(v)$, provided $\partial f_i/\partial u$ $(i = 1, 2, 3)$ do not vanish. We now assume that the u series and v series are orthogonal; then $\partial x/\partial v$, etc., vanish along the contact curve. The series of contact curves lie on a surface. We consider the curve in the region of an ordinary point, at which not all the u-derivatives vanish, and the normal planes of the surface. Let the plane N_1 contain the tangent to the contact curve, and the plane N_2 be perpendicular to N_1 and cut the surface in the curve L, and let ν be the smallest value of n for which the derivatives $\partial^n x/\partial v^n$, $\cdots$ do not all vanish. For the coordinates of the point of intersection of $v + \Delta v = \text{const.}$ and L we have expansions of the form

$$x + \frac{1}{\nu!}\frac{\partial^\nu x}{\partial v^\nu}(\Delta v)^\nu + \cdots; \tag{4}$$

and

$$(\xi - x)\frac{\partial^\nu x}{\partial v^\nu} = 0$$

is the equation of N_1. For odd values of ν, the part of the curve L corresponding to positive values of Δv lies on one side of N_1, and that corresponding to negative values on the other. The contact curve is therefore not a proper envelope. For even values of ν two cases are possible; first, $\Delta v > 0$ may correspond to a different part of the curve L than $\Delta v < 0$, in which case the curve L has a cusp at its point of intersection with the contact curve, which is an edge of regression on the surface; if, however, both signs of Δv correspond to the same point of L, from (4) no conclusions can be drawn. In this case the contact curve is an envelope of the series $v = \text{const.}$ This can be seen by rolling the tangents of a plane curve around a cylinder.

Given the family of surfaces $f(x, y, z, v) = 0$, and the curves

$$f = 0, \quad \partial f/\partial v = 0$$

(characteristics); if the equations

$$f = 0, \quad \partial f/\partial v = 0, \quad \partial^2 f/\partial v^2 = 0$$

have a complete solution, corresponding geometrically to a curve, this curve is a contact curve of the series of characteristics, provided

$$\frac{\partial f}{\partial y} \cdot \frac{\partial^2 f}{\partial z \partial v} - \frac{\partial f}{\partial z} \frac{\partial^2 f}{\partial y \partial v}, \quad \text{etc.},$$

are not all zero. It is not necessarily an envelope or an edge of regression. Monge's "arête de rebroussement" is not entirely justified, as is seen in the example of the circles of curvature and spheres of curvature of a space curve.

8. Dr. Velten's paper will appear in full in the next number of the *Jahresbericht*.

9. Professor Voigt first pointed out a number of serious errors in the current mathematical theory of taxes and tariff, and mentioned the remedy found in his recent book: Mathematische Theorie des Tarifwesens (Jena, 1912), namely, by means of a rational construction of the tariff from a mathematical basis. Certain advantages to both the importer and to the government were explained which would ensue from this procedure.

VIRGIL SNYDER.

SHORTER NOTICES.

Differential and Integral Calculus. By Professor L. S. HULBURT. New York, Longmans, Green, and Co., 1912. xviii + 481 pp. with figures.

THE subtitle of this volume, "An introductory course for colleges and engineering schools," indicates the scope of the author's aims. In view of the numerous elementary text books on the calculus, each enjoying more or less popularity at the present time, it might seem a priori that a newcomer in the field would have difficulty in displaying sufficient individuality to warrant its entrance upon the stage. But no

reader of Professor Hulburt's book will have any difficulty in recognizing a master hand at all stages from the modest preface to the closing chapter on differential equations. In the matter of notation, of selection and arrangement of topics and of kindred points we may continue to expect a wide variance of opinion, but it is obvious that the author has decided views on these matters, and has not hesitated to incorporate them in his book. It may seem rather presumptuous to criticize without trial what such a successful teacher as Professor Hulburt tells us is the result of test and trial in the class room, but as the fond parent is ever prone to regard the idiosyncrasies and peculiarities of his own children as evidences of promise or genius, the critic may at all times be expected to differ on some points. Many teachers for example prefer the symbol d/dx to D_x, which the present text employs, but both notations are so common that even the beginner should be familiar with both. Typographically the book is well done and quite free from errors, although the very frequent use of a double line within brackets, when stating a theorem or definition consisting of two parts, differing only by the substitution of an alternative word or phrase at one or more places, does not add to the attractiveness nor interest of certain pages. The use of very heavy cancellation marks in the algebraic reduction of illustrative examples is quite conspicuous; but it is obvious that these are mere matters of taste or convenience. The double use, in close proximity, of e as the base of natural logarithms and as the eccentricity of a conic, without explicit statement, might lead to confusion. There seems to be no reason for the author's defining the circular functions as the *inverse* trigonometric functions instead of adhering to the well established meaning as found in various books on the theory of functions as well as on the calculus.

The author has maintained clearness without sacrificing rigor, wherever he has essayed a proof. But he has not hesitated to use a theorem without proof where he believes the rigorous proof would not appeal to the ordinary beginning student. As a typical illustration of this we might mention the theorem, "In taking a mixed partial derivative of any order, provided the function be continuous, it is immaterial in what order the differentiations are performed." Among others the general theorems on limits, and the ratio test for the convergence of infinite series are stated without proof.

This omission of proofs which do not enlighten is in accordance with generally accepted sound pedagogy in other courses. Again the author does not hesitate to introduce a new idea by a working definition or statement of a theorem, reserving the rigorous analytic proof for later pages. For example, he begins by defining a function as being discontinuous for real values of the variable "wherever it has a break in its graph." The analytic definition of discontinuity follows later. The author uses the method of limits, as distinguished from the method of rates, and appeals strongly to the geometrical intuition from the start. But at no time does he seem to lose sight of the important truth that a clear conception of the fundamental ideas of the calculus is of more importance than the mere ability to manipulate the formal side, even in applications. For purposes of convenience the volume is divided into six books, each of which is more or less of a unit in itself. The first two constitute a quite complete elementary course in themselves.

Book I, with its seventeen chapters and 175 pages, comprises considerably more than one third of the text and is a complete elementary course in the differential calculus of functions of one variable. The introductory chapters on functions, discontinuities of functions, and limits are models of clearness and should enable any intelligent beginner to start upon his course with an adequate understanding of these rather difficult concepts. The distinction between the *equation* of a graph, and the *function* which the graph represents is made extremely clear. In this book the study of the convexity and concavity of algebraic curves, and of their maxima, minima, and flexes* is taken up before the formulas for differentiating logarithmic and exponential functions are derived. The illustrations and examples in this book are almost entirely geometrical, except in the brief treatment of velocity. Chapters X to XVI are devoted to plane curves, separate chapters being devoted to parametric equations, to polar equations, and to cycloidal curves. Curvature, involutes, and evolutes are treated briefly. The collection of curves is a very interesting one. It may be noted that, despite the amount of space devoted to curves, the subjects of asymptotes

* Professor Hulburt adopts the terms "flex," "flex tangent," etc., instead of the longer "point of inflection," "inflectional tangent," etc. The terms are due to Professor Frank Morley.

and of general methods of finding singularities are not taken up. As these topics throw little added light on the meaning or application of derivatives, the omission is doubtless warranted, and they are rightly reserved for works on higher plane curves or projective geometry, where they may be completely treated. However interesting these chapters are to the mathematical student, it may seem to some teachers that they might be curtailed or, preferably, be reserved in part for some later book, although logically they belong here. There is a demand from the physicists and others in engineering schools that the formal elements of both differential and integral calculus be given to the student as early as possible, and this goal might be approached by deferring these chapters of Book I until after Book II had been studied. Indeterminate forms are concisely treated in the last chapter.

Book II covers 90 pages, of which but 25 are devoted to the technique of formal integration, the major stress being on integration by the aid of tables. Of the ordinary special methods of integration, but three are explained in detail—algebraic substitution, trigonometric substitution, and integration by parts. The reduction formulas have been deservedly ejected from their time honored seat, and do not appear in the volume at all, while the integration of rational fractions is reserved for a chapter in a later book. Nineteen pages are devoted to the applications of integration in kinematics, so that this topic is treated with some detail, and accomplishes well its obvious purpose of fixing the significance of the constants of integration and of familiarizing the student with typical applications of the indefinite integral. The remainder of this book is devoted to the simple applications of the definite integral,—calculating the lengths of arcs of plane curves, the areas under them, and the surfaces and volumes of solids of revolution. The clearness of statement in the two chapters on the definite integral deserves special note as the student is prone to swallow this part of the theory on faith and to trust to his works in solving examples to bring about his salvation. Yet at the very beginning the author draws a conclusion after what seems to be the least convincing argument in the entire book. After showing that x^3 and $x^3 + c$ are both integrals of $3x^2$, he immediately states, "Hence the *most general* (italics mine) integral of $3x^2$ is $x^3 + c$." It would seem to the reviewer that a brief table of integrals might

well have been included in the volume as a matter of convenience if not of necessity, since the author specifically aims to teach the students to use a table skilfully. The references to paragraphs in an alien book are at times slightly confusing, and it seems, in general, unwise to refer to paragraph numbers which may be changed at the whim of another. Many teachers will regret the absence of a discussion of methods of approximate integration.

Book III, which is devoted to an introduction to analytic geometry of three dimensions, will be warmly welcomed by teachers who find it necessary to take up the calculus without a formal course in solid analytics. The essentials, everything required in the applications of the calculus, are given compactly (some will say too compactly) but clearly and comprehensively in the compass of 20 pages. The teacher whose students have covered this subject matter will find it convenient for review and reference, whereas others will probably find no serious break in continuity since the necessity for a three-dimensional coordinate system is presented by the nature of the problems to be attacked. Volumes of simpler solids and areas of surfaces are treated in this book by single integration only.

Book IV, of 55 pages, is a brief calculus of functions of several real variables. Partial and total derivatives with their geometrical interpretation and application, and multiple integrals with their application to areas, volumes, surfaces, and centers of mass are the principal topics treated. The theory of partial and total derivatives and differentials is treated in considerable detail and one sees a trace of humor in the closing remark, "We repeat . . . that small advantage accrues from the use of differentials. They are relics of the early days of the calculus, and *as the reader is probably now ready to admit* (italics mine) constitute for the beginner an obstacle to the understanding of the calculus rather than an aid. Nevertheless they are in almost universal use and it is therefore necessary that the student of mathematics become thoroughly familiar with them." Teachers may differ as to the utility of this relic. Such topics as work, fluid pressure, and center of pressure are omitted. It is doubtless true that these subjects strengthen very slightly the grasp of the student on the principles and methods of the calculus, as the mathematics is lost sight of in struggling with the ideas of physics and

mechanics involved. The moment of inertia integral is also not introduced at all, and center of mass is defined as an integral, without any physical interpretation or derivation.

Book V, of 72 pages, is devoted to a few special topics, chiefly to Taylor's and MacLaurin's theorems, De Moivre's theorem, hyperbolic functions, integration of rational fractions, and envelopes. Many teachers doubtless prefer to take up some of these topics earlier, to which the author would certainly not object, as they are collected in this rather heterogeneous book for convenience of omission or inclusion. The devotion of a few pages to such an important formula as De Moivre's theorem seems justified, although it is omitted from most texts in common use. The treatment of Taylor's theorem possesses peculiar clearness and charm. Based primarily on the extension of the law of the mean, the development is first in the finite form where the remainder is retained and discussed, and then in the infinite form, applying of course only when the series is convergent,—a treatment which certainly tends to clearness. The caustic is a leading example of an envelope.

Book VI is entitled "An Introduction to Ordinary Differential Equations," and the subject is admirably treated within the limits of 30 pages. It is vastly more than a mere tabulation of methods of solving certain types of differential equations. A half dozen pages are devoted to clearing up the ideas involved in such terms as "solution of a differential equation," "particular integral," "complete primitive," and the like. Then all classes of equations ordinarily met by students in applied mathematics are treated, with illustrative examples. Clairaut's equation is given largely as a curiosity. It would seem to the reviewer that it would have been worth while to explain in somewhat more detail the special method of solving linear differential equations of the first degree when the right hand member contains a term which is also a term of the complementary function, as it would throw added light on the whole topic. The remarks by the author on this subject (page 446) are scarcely as full as the student will need to make them truly enlightening.

Professor Hulburt has, as is clear from the above outline, attacked the problem of producing a text book suitable for both colleges and engineering schools with characteristic vigor. Whether the course be one to finish off the mathe-

matical studies of the college student, or to mark the completion of formal mathematics for the prospective engineer, or be but the stepping stone to further advanced study for the mathematical student, the arrangement and the presentation in this volume will materially aid the teacher and student. There is sufficient of the spirit of research and rigorous analysis to meet the demands of the last class, while the necessities of the others have not been neglected. The lists of problems are of sufficient variety and extent to meet all ordinary requirements. It will be quite clear to the careful reader that both in illustrations and in exercises, the author has avoided those examples which, however interesting in themselves, present their greatest difficulty or interest because of the dynamics, physics, or other science involved, and beyond that shed little or no new light on the methods of the calculus. In fact the field of dynamics is entirely avoided. On the other hand the author does not attempt to conceal or minimize the real difficulties of calculus by employing trivial examples. The answers to all exercises are conveniently assembled at the end of the text. The index is exceptionally complete. There is no doubt but that this text will be a valued addition to the teacher's library and will find a deserved admission into many class rooms.

D. D. LEIB.

Theorie der Zahlenreihen und der Reihengleichungen. By ANDREAS VOIGT. Leipzig, Göschen, 1911. viii + 133 pp.

THE two fundamental ideas which underlie this work are the following:

1. Instead of considering a number as isolated, one may think of it as belonging to a sequence. Thus the question as to whether b is divisible by a is the question as to whether b belongs to the sequence $\cdots, -2a, -a, 0, a, 2a, \cdots$.

2. Instead of expressing an integer N as a polynomial in x of the form

$$N = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

one may write it in either of the forms

$$N = b_0x(x-1)\cdots(x-n+1) + b_1x(x-1)\cdots(x-n+2) + \cdots + b_{n-1}x + b_n,$$

$$N = c_0x(x+1)\cdots(x+n-1) + c_1x(x+1)\cdots(x+n-2) + \cdots + c_{n-1}x + c_n.$$

These ideas are not new; but the author has sought to make a systematic use of them in developing a theory of number sequences. Two such fundamental number sequences are considered, each of which is a generalization of a sequence of binomial coefficients. Several important sets of numbers can be expressed in terms of these fundamental sequences, as for instance the set of figurate numbers in which the rth term of the nth row equals the sum of the first r terms of the $(n-1)$th row, the first row being 1, 0, 0, $\cdots$. A general theory of the two fundamental sequences is developed and the results are applied to several questions in number theory; as, for instance, the solution of congruences and diophantine equations. The methods employed are such that they cannot be explained briefly.

R. D. CARMICHAEL.

NOTES.

BEGINNING with volume 20 (1913), the *American Mathematical Monthly* will be in charge of an editorial board composed of representatives of nine supporting institutions, together with Professor B. F. FINKEL, the founder of the journal and editor since its inception in 1894. The contributing institutions are Colorado College and the Universities of Chicago, Illinois, Missouri, Minnesota, Nebraska, Kansas, Indiana, and Iowa. The editorial representatives are Professors FLORIAN CAJORI, H. E. SLAUGHT, G. A. MILLER, E. R. HEDRICK, W. H. BUSSEY, W. C. BRENKE, C. H. ASHTON, R. D. CARMICHAEL, and R. P. BAKER. The managing editor is Professor Slaught.

It will be the editorial policy of the *Monthly* to make a strong appeal to the great body of teachers in the collegiate and advanced secondary fields, not only directing attention to questions of improvement in teaching but also fostering the development of the scientific spirit among large numbers who are not now reached by the more highly technical journals. The publication of original papers will be continued, but greater attention than heretofore will be given to pedagogical and historical questions of interest and value to teachers of collegiate mathematics. An index of volumes 1–19 will soon be issued.

THE annual meeting of the London mathematical society was held on November 14, 1912. The following papers were presented: By H. F. BAKER, presidential address, "Recent advances in the theory of surfaces"; by A. B. GRIEVE, "Some properties of cubic surfaces"; by W. H. YOUNG, "The determination of the summability of a function by means of its Fourier constants"; by W. BURNSIDE, "Groups of linear substitutions of finite order which possess quadratic invariants"; by J. B. HOLT, "The irreducibility of Legendre's polynomials"; by E. W. HOBSON, "The representation of a summable function by means of a series of finite polynomials"; by E. CUNNINGHAM, "The theory of functions of real vectors."

Professor A. E. H. LOVE was chosen president of the society for the coming year.

AT the meeting of the Edinburgh mathematical society on November 8 the following papers were read: By H. S. CARSLAW, "Integral equations and the determination of Green's function in the theory of potential"; by F. E. EDWARDS, "On a certain infinite expansion"; by Mr. SWAMINARAYAN, "A determinantal proof of Ptolemy's theorem."

THE third annual meeting of the Schweizerische Mathematische Gesellschaft was held at Altdorf during the week beginning September 10, 1912, under the presidency of Professor R. VON FUETER and in affiliation with the annual meeting of the Swiss association of science. Professor H. FEHR was chosen president for the ensuing year. The following papers were presented at the meeting: "Ueber die Einteilung der Idealklassen in Geschlechter," by R. VON FUETER; "Ueber Ponceletsche Polygone," by — BÜTZBERGER; "Projektiver Beweis der absoluten Parallelkonstruktion von Lobatschefskij," by M. GROSSMANN; "Sur quelques problèmes concernant le jeu de trente et quarante," by D. MIRIMANOFF; "Ueber Gruppen algebraischer Funktionen," by O. SPIESS; "Sur les singularités des surfaces," by G. DUMAS; "Unicité du développement d'une fonction en série de polynômes de Legendre et expression analytique des coefficients de ce développement," by M. PLANCHEREL; "Nouveaux modèles de mouvement pour l'enseignement de la géométrie," by J. ANDRADE; "Kinematische Untersuchung," by E. MEISSNER; "Ueber eine besondere konforme rationale Transformation

in der Ebene," by A. EMCH; "Continuité et discontinuité" and "Sur le mouvement le plus général d'un fluid dans l'espace," by R. DE SAUSSURE; "Der Stand der Herausgabe der Werke Leonhard Eulers," by F. RUDIO; "L'état des travaux de la commission internationale de l'enseignement mathématique et de la sous-commission suisse," by H. FEHR.

THE firm of Martin Schilling in Leipzig has announced two new series of models: one is a combined gyroscope and pendulum, and the other consists of three cardboard models of the Bessel functions with complex argument.

THE Carnegie Institution of Washingon, of Washington, D. C., announces the following books in press: H. W. STAGER, "A Sylow factor table for the first twelve thousand numbers, giving the possible number of subgroups under Sylow's theorem of a group of given order between the limits of 0 and 12,000"; D. N. LEHMER, "Tables giving a complete list of the prime numbers between the limits 1 and 10,006,721."

THE following list, compiled from the "Jahres-Verzeichnis der an den Deutschen Universitäten erschienenen Schriften," volume 26, comprises the list of successful candidates for doctorates in mathematics in the German universities for the academic year 1910–11. The list is incomplete in omitting the names of candidates whose dissertations were unpublished before the volume appeared, early in 1912. The title of the dissertation, number of pages, date of publication, and name of the chairman of the examining committee are added.

Berlin.

MÜNTZ, CH.: "Zum Randwertproblem der partiellen Differentialgleichung der Minimalflächen." 34 pp. Oct. 1, 1910. Schwarz.

REMAK, R.: "Ueber die Zerlegung der endlichen Gruppen in direkte unzerlegbare Faktoren." 20 pp. Feb. 25, 1911. Frobenius, Schwarz.

STEINBACHER, F.: "Abelsche Körper als Kreisteilungskörper." 16 pp. Dec. 14, 1910. Frobenius, Schwarz.

Bonn.

BRÜES, M.: "Zur Theorie der desmischen Flächen vierter Ordnung." v + 48 pp. Nov. 23, 1910. Study.

Breslau.

KOBER, H.: "Konjugierte kinetische Brennpunkte." 77 pp. Nov. 23, 1910. Kneser.

Freiburg.

MONTFORT, P.: "Die Auflösung der numerischen Gleichungen nach Fourier." 81 pp. 1911. Lüroth.

Giessen.

CHAMBRÉ, A.: "Darstellung von Faktoren ganzer Funktionen durch Kovarianten." 36 pp. Nov. 7, 1910. Pasch.

DRESCHER, E.: "Ueber geometrische Darstellung von Gruppen." 22 pp. Feb. 6, 1911. Netto.

SCHREITER, FR.: "Ueber das kombinatorische Produkt von vier Kollineationen im Raum und die Apolarität kollinearer Verwandschaften auf allen Stufen." 60 pp. April 12, 1911. Pasch.

SEEMAN, H.: "Projektive Verallgemeinerung metrischer Begriffe." 24 pp. Sept. 21, 1910. Pasch.

THAER, FR.: "Analytische Beiträge zur Lehre vom Kegelschnittsystem ($3p$, $1l$)." 28 pp. April 15, 1911. Pasch.

VAERTING, MARIE: "Zur Transformation der vielfachen Integrale." 35 pp. Oct. 22, 1910. Pasch.

WOLFF, G.: "Ueber Kollineationen in der Ebene." 60 pp. Sept. 21, 1910. Pasch.

Göttingen.

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HAUPT, O.: "Untersuchungen über Oszillationstheoreme." 50 pp. June 27, 1911. Rost.

THE philosophical faculty of the University of Göttingen announces the following problem for the Beneke prize for 1915:

A complete discussion of the problem of viscous motion from the standpoint of hydrodynamics. The discussion, either theoretical, experimental or both, should advance our present knowledge of the resistance offered to the motion of a solid in a fluid and the resistance to fluid motion in tubes and canals. Competing memoirs may be written in any modern language, should be signed with a motto, and must be received by the faculty by August 31, 1914. The major prize will be M. 1,700 and the minor M. 680.

THE Copley medal of the Royal Society has been awarded to Professor F. KLEIN "for his researches in mathematics."

DR. K. BARTEL has been appointed docent in geometry in the technical school at Lemberg.

DR. E. HECKE has been appointed docent in pure mathematics in the University of Göttingen.

DR. KADERAVEK has been appointed docent in synthetic geometry in the Bohemian technical school of Prague.

AT the University of Munich the following changes have been made: Dr. F. HARTOGS has been promoted to an associate professorship; Dr. A. ROSENTHAL has been appointed docent; Dr. H. DINGLER has been appointed docent in the history and teaching of mathematics.

Dr. J. V. E. WESTFALL, of the Equitable life assurance society, has been promoted to the position of acting third vice-president. Dr. Westfall was formerly assistant professor of mathematics in the University of Iowa.

THE announcement in the Notes of the December BULLETIN that Mr. G. H. ALBRIGHT had been appointed exchange professor to Harvard University proves to be inexact.

AT the close of the present academic year Professor W. E. BYERLY, of Harvard University, will retire from active service with the title of professor emeritus.

THE death is announced of Professor G. LANDSBERG, of the University of Kiel, on September 14, 1912, at the age of 47 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMSLER (A.) und RUDIO (F.). J. Amsler-Laffon. Mit Bildnis. Zürich, Zürcher und Furrer, 1912.

ARCAIS (F. D'). Analisi infinitesimale. 3a edizione, con modificazioni ed aggiunte. Volume I. Padova, Draghi, 1912. 8vo. 12+791 pp. L. 12.50

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BJÖRSTEDT (P. T.). De exakta π-värdet jämte en ny polygonteori. Stockholm, 1912. 8vo. 92 pp. M. 3.00

BÖGER (R.). Symmetrische Involutionen und fokal getrennte Geraden. (Progr.) Beilstein, 1912. 56 pp.

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FESTSCHRIFT Heinrich Weber. Zu seinem 70. Geburtstage am 5. März 1912. Gewidmet von seinen Freunden und Schülern. Leipzig, Teubner, 1912. 8vo. 8+500 pp. M. 24.00

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